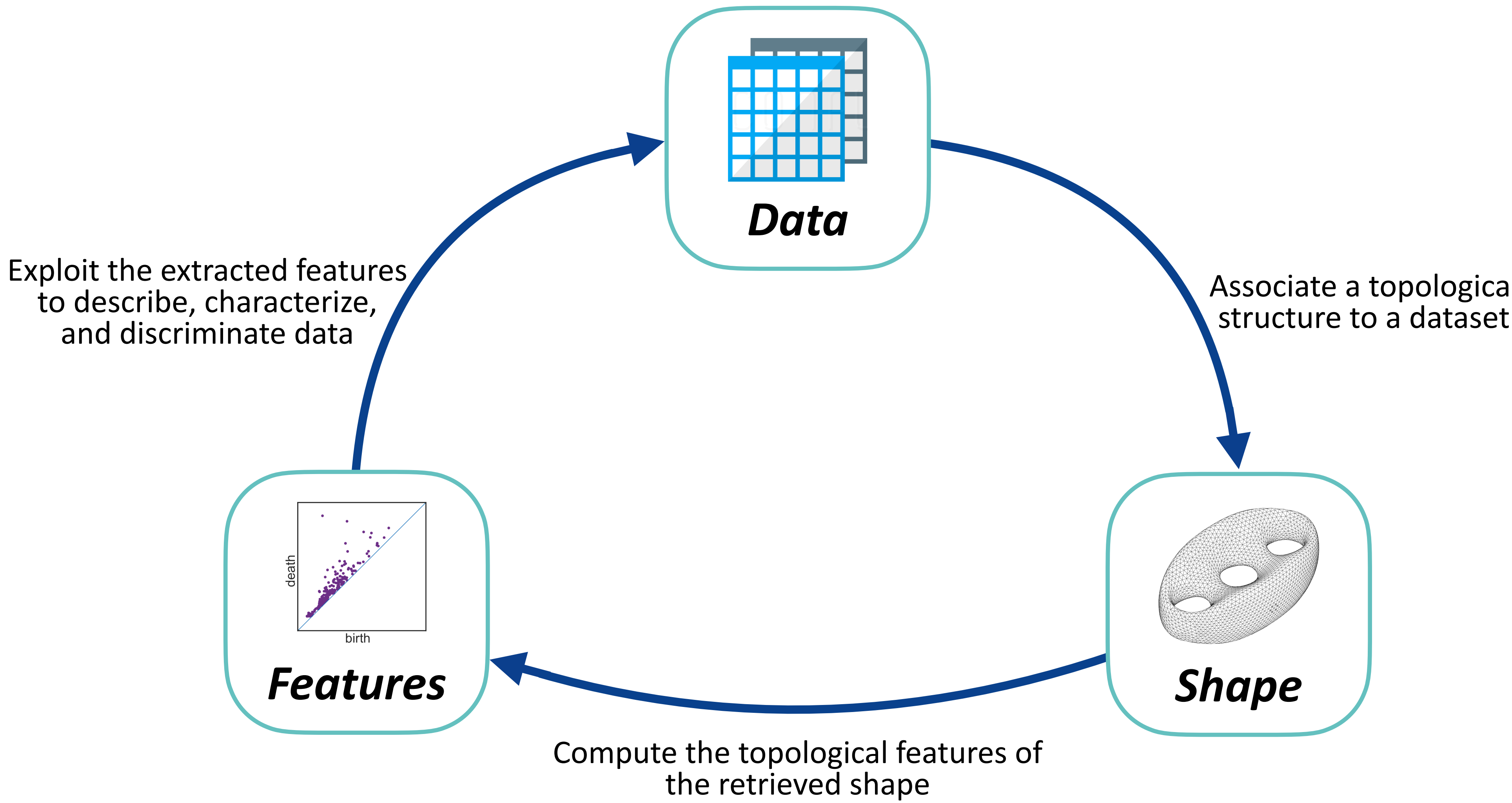
Topological Data Analysis

Homology & Persistence

Ulderico Fugacci

CNR - IMATI

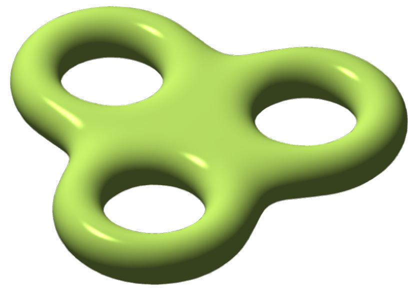
Topological Data Analysis



Homology & Persistence

Homology:

A topological invariant detecting the *"holes"* of a shape

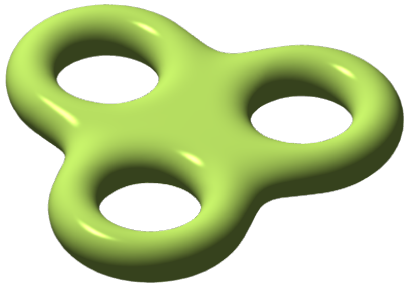


$$H_k(K) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z}^6 & \text{for } k = 1 \\ \mathbb{Z} & \text{for } k = 2 \end{cases}$$

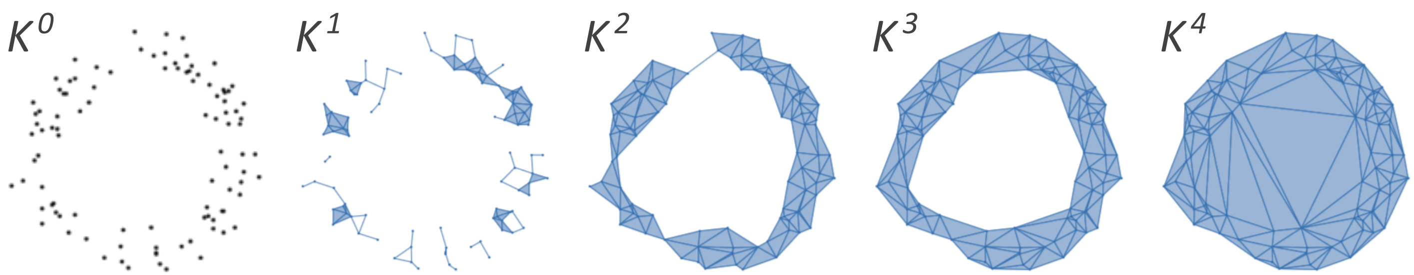
Homology & Persistence

Homology:

A topological invariant detecting the ***"holes"*** of a shape



$$H_k(K) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z}^6 & \text{for } k = 1 \\ \mathbb{Z} & \text{for } k = 2 \end{cases}$$



Persistent Homology:

Captures the ***changes in homology*** during a filtration

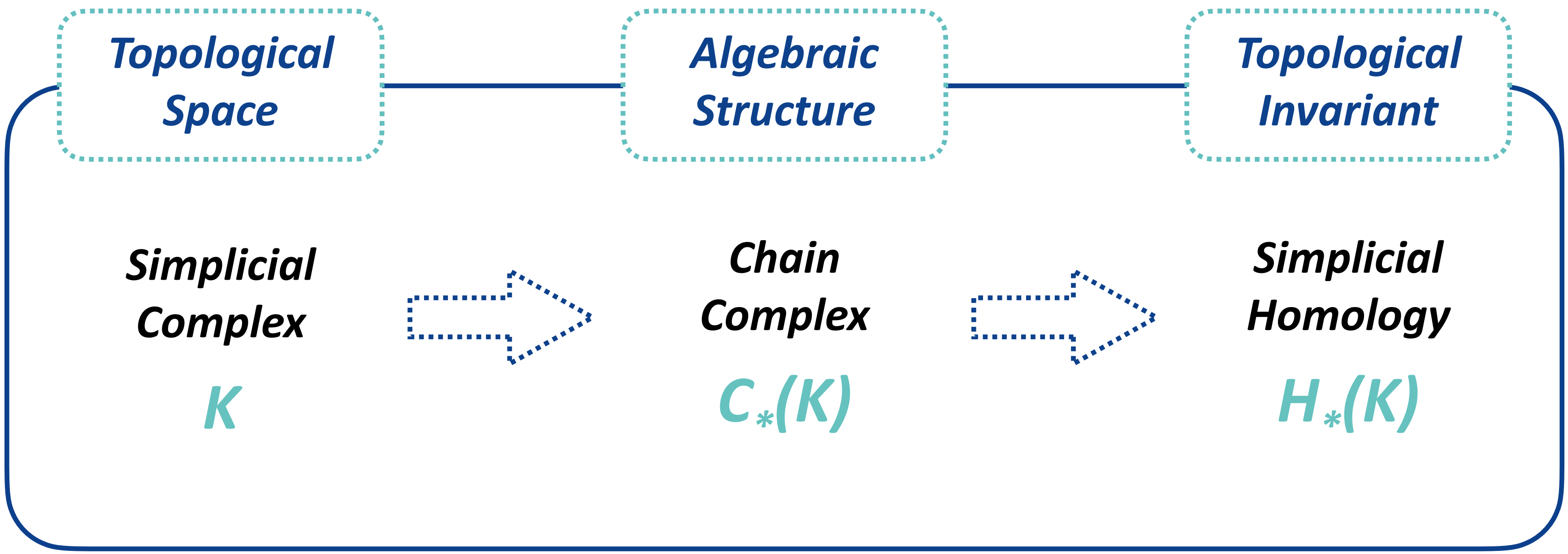
Homology & Persistence

- *Chain Complexes and Simplicial Homology*
- *Filtrations Persistent Homology*

Homology & Persistence

- ***Chain Complexes and Simplicial Homology***
- *Filtrations and Persistent Homology*

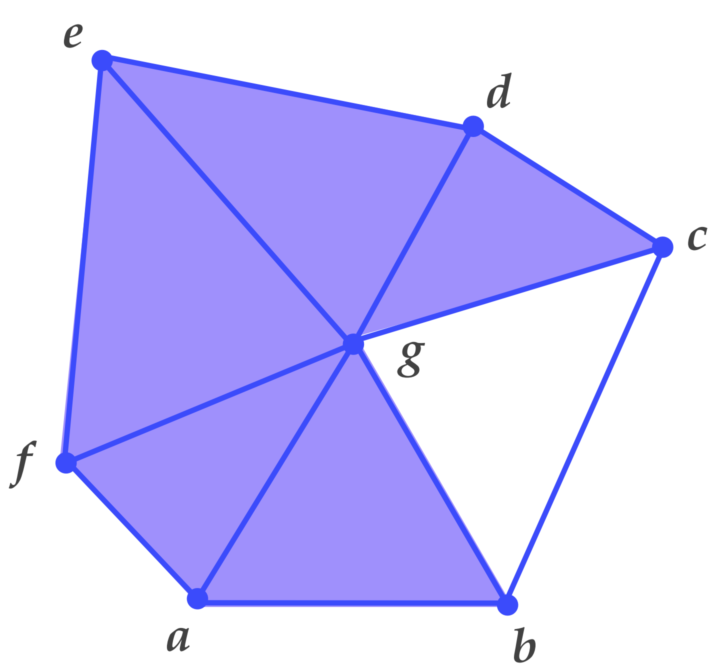
Simplicial Homology



Simplicial Homology

Given a simplicial complex K,

- a ***k-chain*** is a formal sum *(with $\mathbb{Z}_2$ coefficients)* of k-simplices of K



Examples:

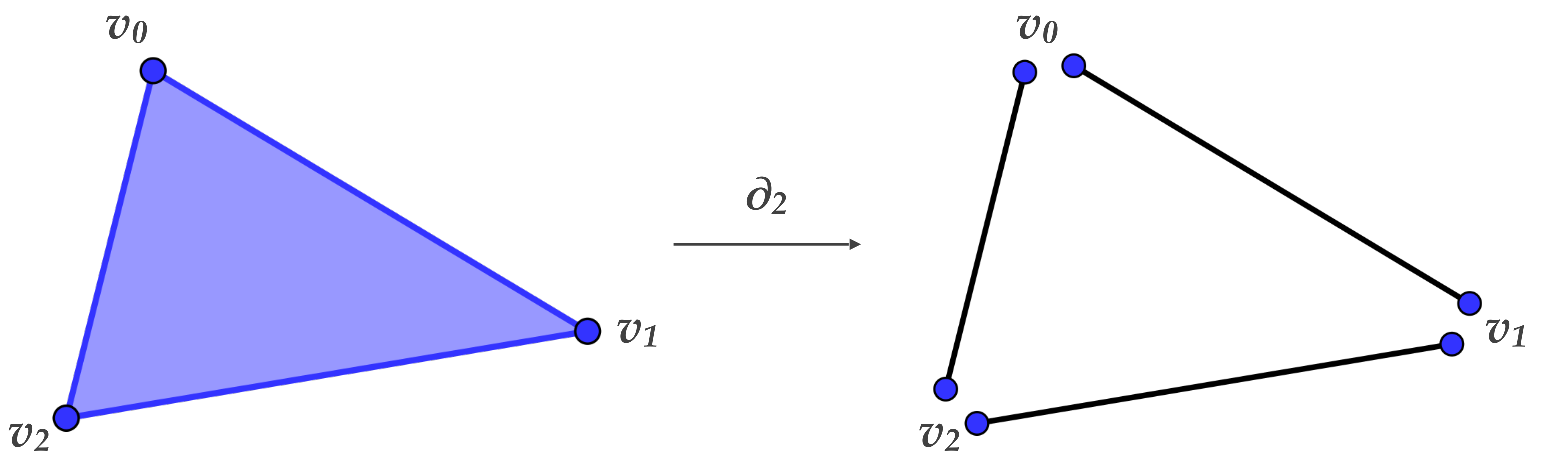
- $a + b + e$ is a 0-chain
- $fg + dg + de + eg$ is a 1-chain
- $abg + afg$ is a 2-chain

Simplicial Homology

The ***chain complex*** $\boldsymbol{C_*(K)}$ ***associated with K*** consists of:

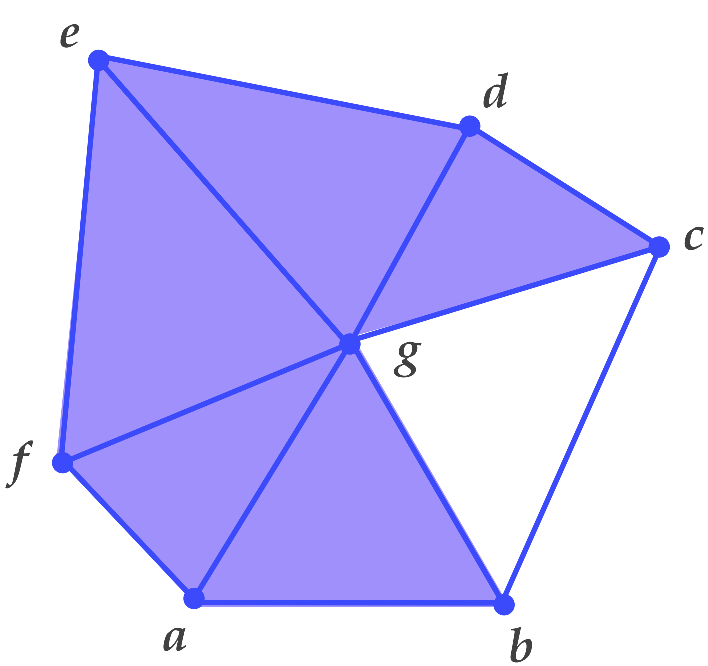
- a collection $\{C_k(K)\}_{k\in\mathbb{Z}}$ of vector spaces where $\mathbf{C_k(K)}$ is the ***group of the k-chains*** of K
- a collection $\{\partial_k\}_{k\in\mathbb{Z}}$ of linear maps where the ***boundary map*** $\boldsymbol{\partial_k : C_k(K) \longrightarrow C_{k-1}(K)}$ is defined by

$$\partial_k(v_0 \cdots v_k) := \sum_{i=0}^{k} v_0 \cdots \hat{v}_i \cdots v_k$$



Simplicial Homology

Examples:



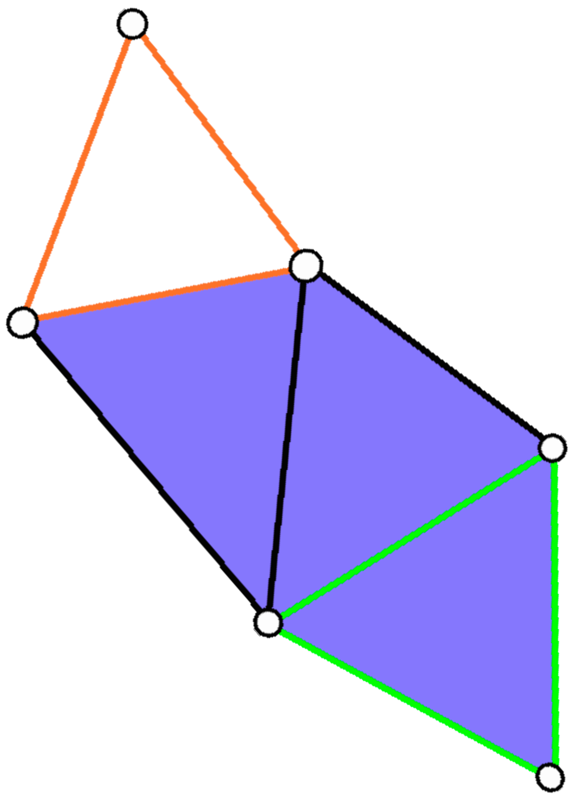
- $\partial_1(\, ab \,) = a + b$
- $\partial_1(\, ab + bc \,) = a + 2b + c = a + c$
- $\partial_2(\, afg + efg \,) = af + ag + 2fg + ef + eg =$
  $= af + ag + ef + eg$
- $\partial_1(\, af + ag + ef + eg \,) =$
  $= 2a + 2f + 2g + 2e = 0$

Simplicial Homology

Properties:

- For $k < 0$ or $k > dim(K)$, $C_k(K)$ is the ***null group***
- For $k \leq 0$ or $k > dim(K)$, ∂_k is the ***null map***
- For any $k \in \mathbb{Z}$, $\partial_k \circ \partial_{k+1} = 0$
- For any $k \in \mathbb{Z}$, $Im(\partial_{k+1}) \subseteq Ker(\partial_k)$

Simplicial Homology



Definition:

A k-chain c is called:

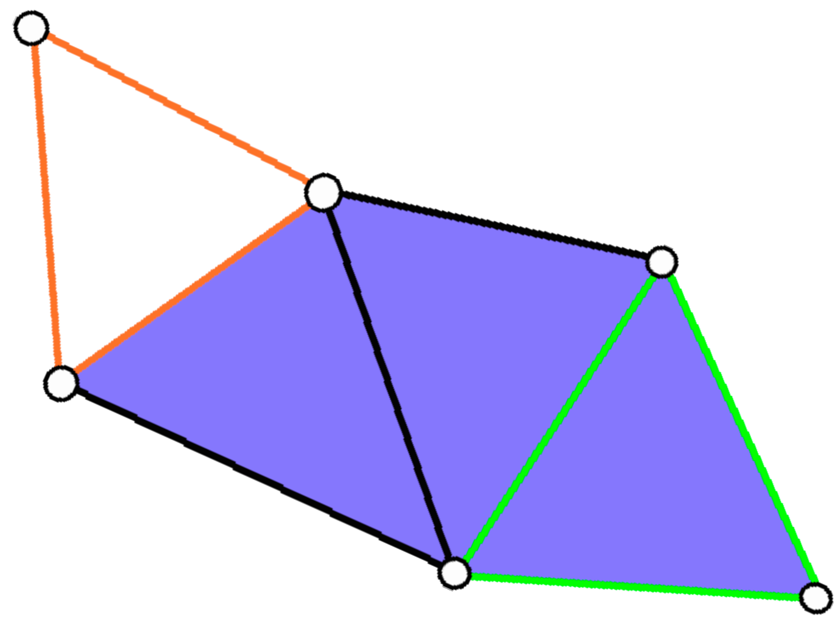
- ***k-cycle*** if $c \in \mathrm{Ker}(\partial_k)$
- ***k-boundary*** if $c \in \mathrm{Im}(\partial_{k+1})$

Each k-boundary is a k-cycle

Simplicial Homology

Given a simplicial complex K, the ***k-homology group*** ***$H_k(K)$*** of K is defined as

$$H_k(K) := Z_k(K)/B_k(K)$$

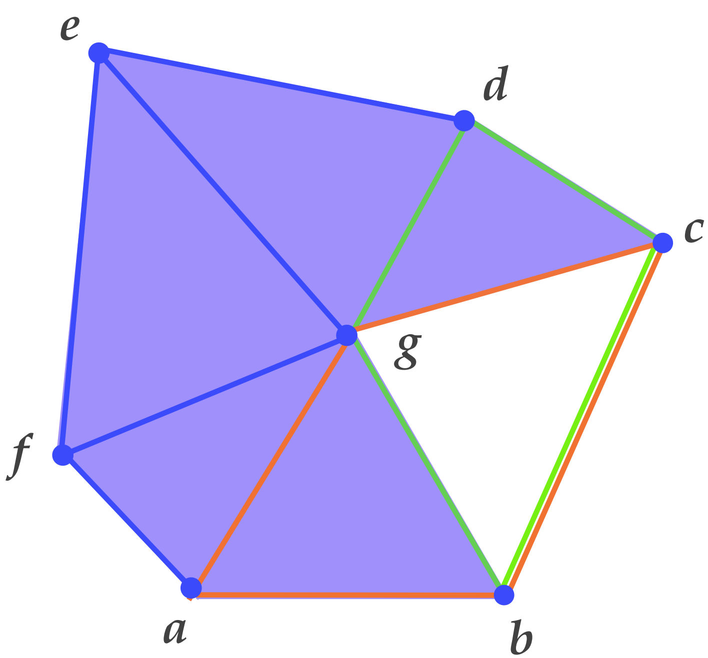


where:

- ***$Z_k(K)$*** is the ***group of k-cycles*** of K
- ***$B_k(K)$*** is the ***group of k-boundaries*** of K

Simplicial Homology

$H_k(K)$ partitions the k-cycles into equivalence classes called ***homology classes***



Definition:

Two k-cycles are said ***homologous*** if they belong to the same homology class or, equivalently, ***if their difference is a k-boundary***

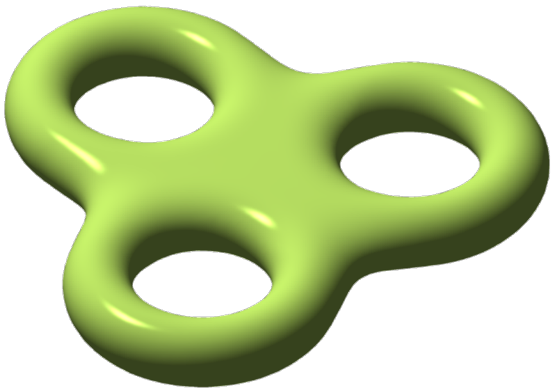
ab+ag+bc+cg is homologous to bc+bg+cd+dg

Simplicial Homology

Theorem:

Each homology group can be expressed as

$$H_k(K) \cong (\mathbb{Z}_2)^{\beta_k}$$



$$H_k(K) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ (\mathbb{Z}_2)^6 & \text{for } k = 1 \\ \mathbb{Z}_2 & \text{for } k = 2 \end{cases}$$

β_k is called the ***k^{th} Betti number*** of K

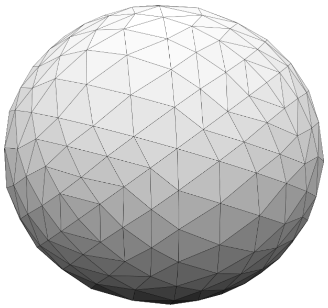
Simplicial Homology

Examples:

- ***point P***

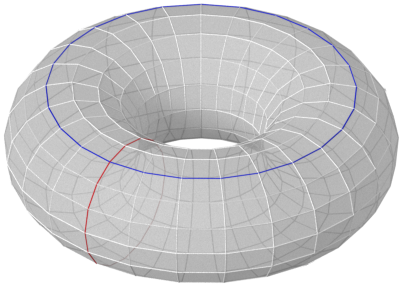
$$\beta_k(P) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

- ***n-dimensional sphere S^n***



$$\beta_k(S^n) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } 0 < k < n \\ 1 & \text{for } k = n \\ 0 & \text{for } k > n \end{cases}$$

- ***torus T***



$$\beta_k(T) = \begin{cases} 1 & \text{for } k = 0 \\ 2 & \text{for } k = 1 \\ 1 & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}$$

Simplicial Homology

Homology groups can be defined ***in a more general way*** by choosing coefficients in $\mathbb{Z}$

Theorem:

Each homology group can be expressed as

$$H_k(K;\mathbb{Z}) \cong \mathbb{Z}^{\beta_k}\langle c_1,\ldots,c_{\beta_k}\rangle \oplus \mathbb{Z}_{\lambda_1}\langle c'_1\rangle \oplus \cdots \oplus \mathbb{Z}_{\lambda_{p_k}}\langle c'_{p_k}\rangle$$

with $\lambda_{i+1} \mid \lambda_i$

We call:

- β_k, the ***k^{th} Betti number*** of K
- $\lambda_1,\ldots,\lambda_{p_k}$, the ***torsion coefficients*** of K
- $c_1,\ldots,c_{\beta_k},c'_1,\ldots,c'_{p_k}$, the ***homology generators*** of K

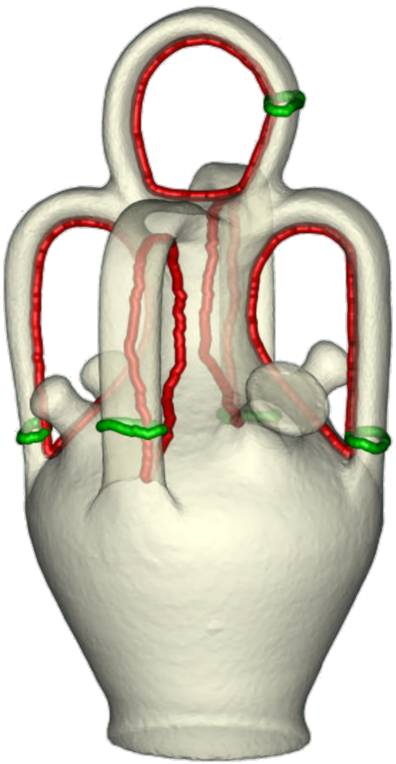


Image from [Dey et al. 2008]

Simplicial Homology

Working with coefficients in $\mathbb{Z}$:

*Up to isomorphism, the **Betti numbers** and the **torsion coefficients** of K*
***completely characterize** the **homology groups** of K*

Working with coefficients in a field $\mathbb{F}$:

*Up to isomorphism, the **Betti numbers** of K*
***completely characterize** the **homology groups** of K*

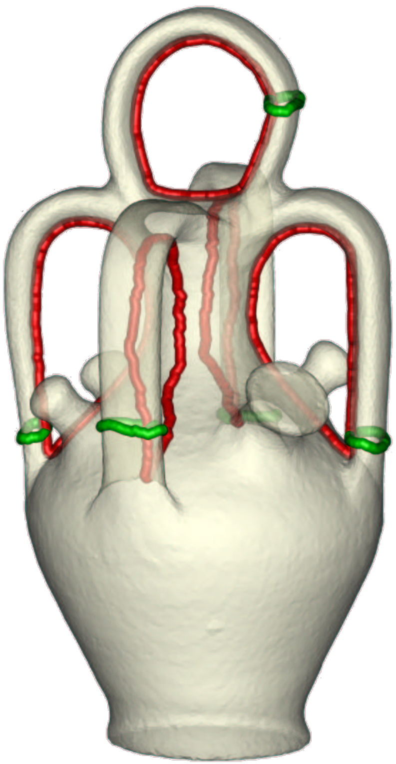
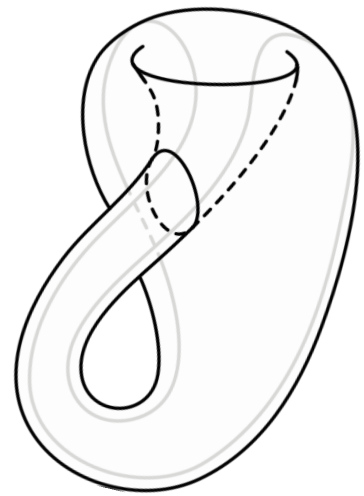


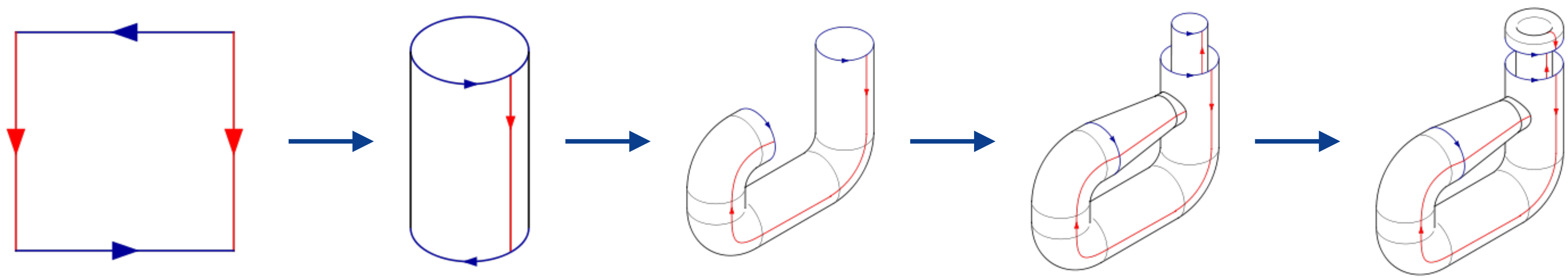
Image from [Dey et al. 2008]

Simplicial Homology

Example:

The ***Klein bottle K*** *is a non-orientable 2-dimensional manifold embeddable in* $\mathbb{R}^4$ *which can be built from a unit square by the following construction*





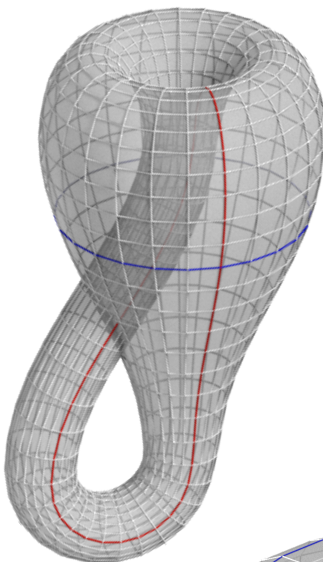
Simplicial Homology

Example:

By considering $\mathbb{Z}$ as coefficient group,

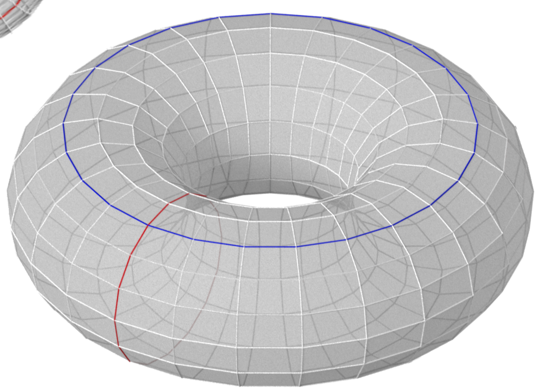
K has the following homology groups

$$H_k(K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } k = 1 \\ 0 & \text{for } k \geq 2 \end{cases}$$



So, it can be distinguished from a torus T

$$H_k(T;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z}^2 & \text{for } k = 1 \\ \mathbb{Z} & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}$$

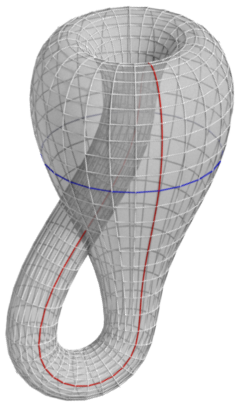


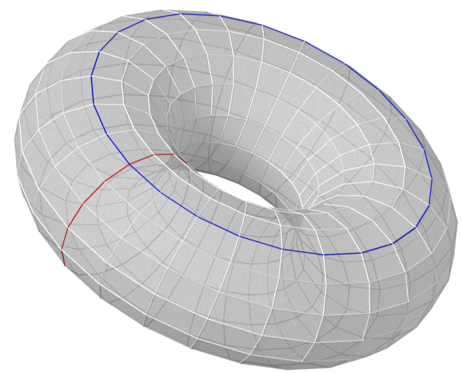
Simplicial Homology

Example:

By considering $\mathbb{Z}_2$ as coefficient group,

the Klein bottle K and the torus T have isomorphic homology groups





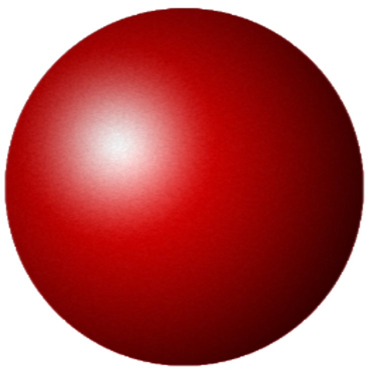
$$H_k(K;\mathbb{Z}_2) \cong \left\{ \begin{array}{ll} \mathbb{Z}_2 & \text{for } k=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } k=1 \\ \mathbb{Z}_2 & \text{for } k=2 \\ 0 & \text{for } k>2 \end{array} \right\} \cong H_k(T;\mathbb{Z}_2)$$

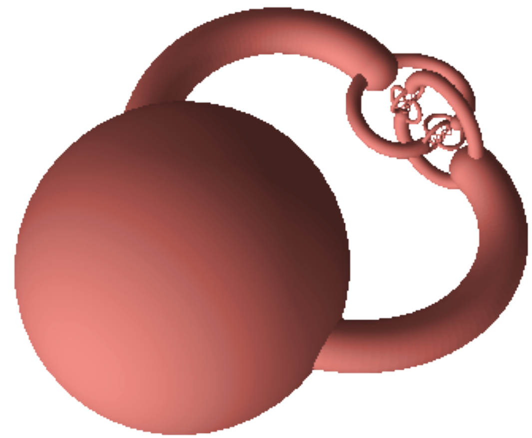
Homology & Persistence

- *Chain Complexes and Simplicial Homology*
- ***Filtrations and Persistent Homology***

Persistent Homology

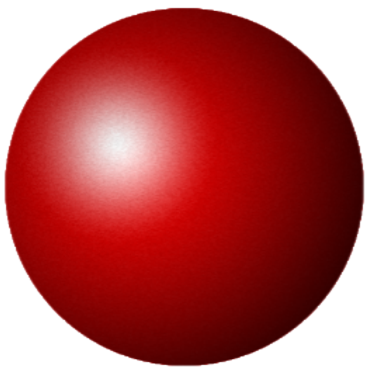
- *Do they have the same shape?*

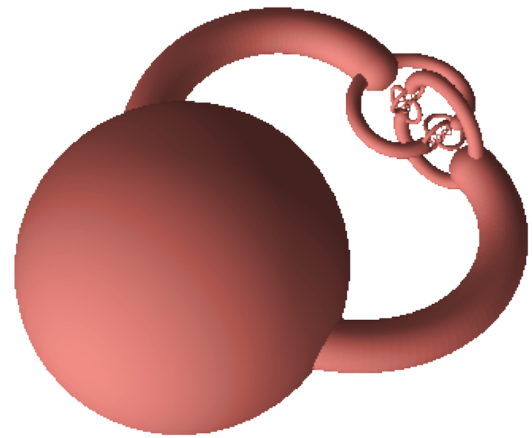




Persistent Homology

- *Do they have the same shape?*

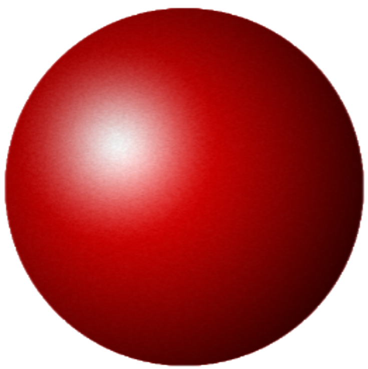


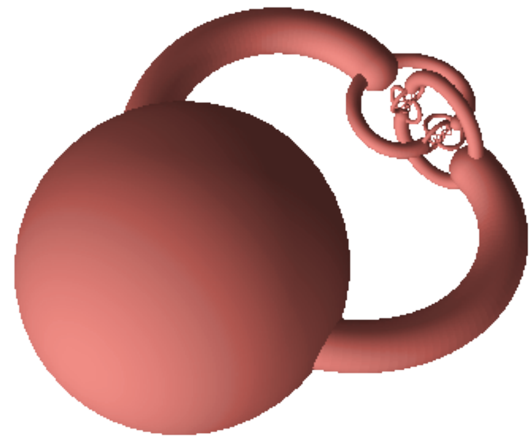


In Practice?	In Theory?

Persistent Homology

✦ ***Do they have the same shape?***





In Practice?	**In Theory?**
	
	They are ***homeomorphic***

Persistent Homology

- *Do they have the same shape?*

Persistent Homology

- ***Do they have the same shape?***

In Practice?

In Theory?

Persistent Homology

- ***Do they have the same shape?***

In Practice?

In Theory?

They are ***not homeomorphic***

Persistent Homology

In a Nutshell:

Persistent homology allows for **describing the changes in the shape** of an evolving object

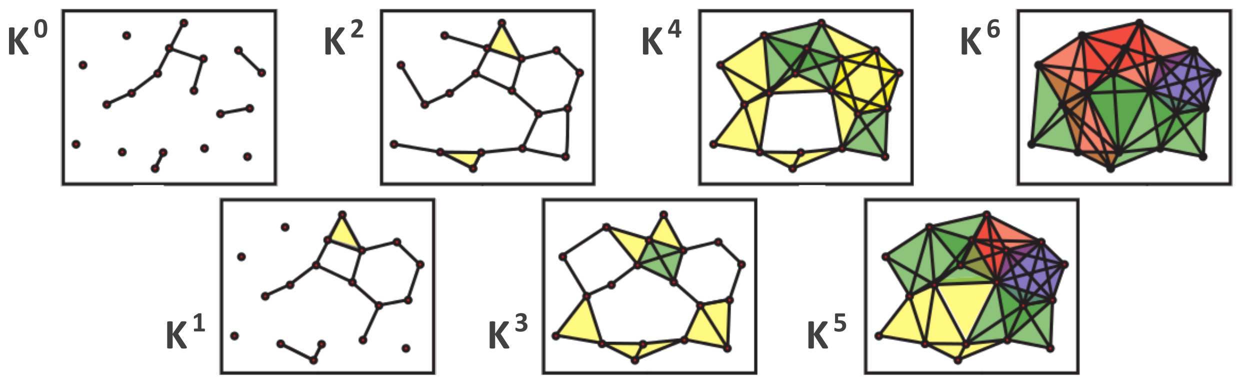


Image from [Ghrist 2008]

Persistent Homology

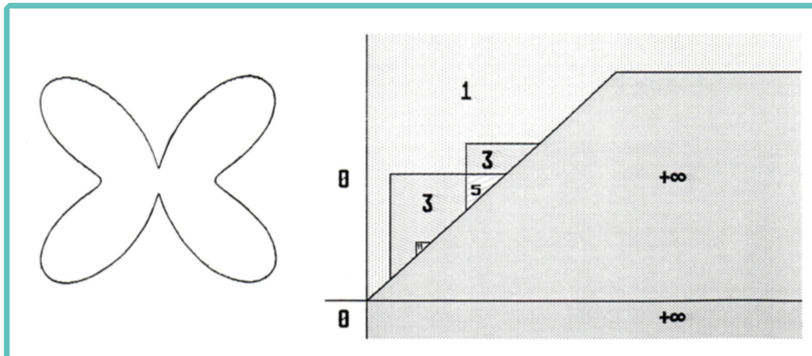
An Evolving Notion:

1990

Frosini

Size Functions:

- ***Estimation of natural pseudo-distance*** between shapes endowed with a function *f*
- Tracking of the ***connected components*** of a shape along its evolution induced by *f*



Actually, this coincides with ***persistent homology in degree 0***

Image from [Frosini 1992]

Persistent Homology

An Evolving Notion:

1990 — Frosini

1994 — Delfinado, Edelsbrunner

Incremental Algorithm for Betti Numbers:

- Introduction of the notion of ***filtration***
- De facto computation of ***persistence pairs***

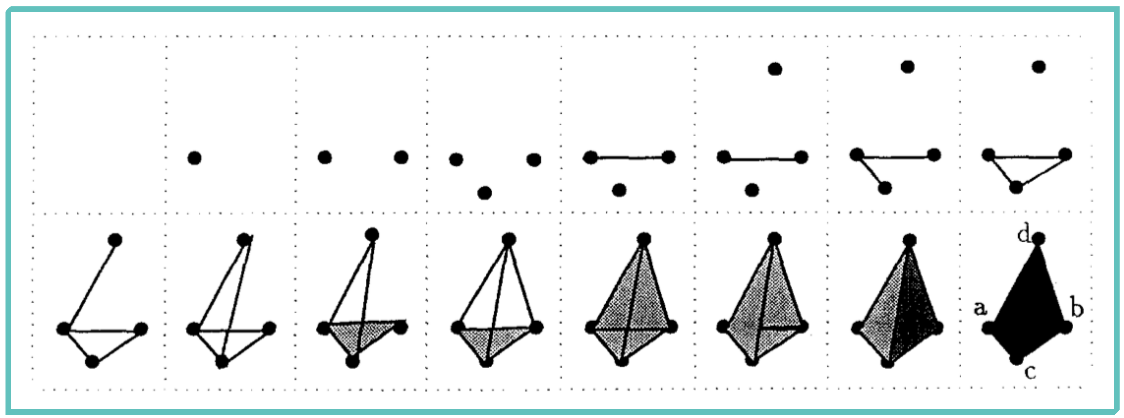


Image from [Delfinado, Edelsbrunner 1995]

Persistent Homology

An Evolving Notion:

1990 — Frosini

1994 — Delfinado, Edelsbrunner

1999 — Robins

Homology from Finite Approximations:

- ***Extrapolation of the homology*** of a metric space from a ***finite point-set approximation***
- Introduction of ***persistent Betti numbers***

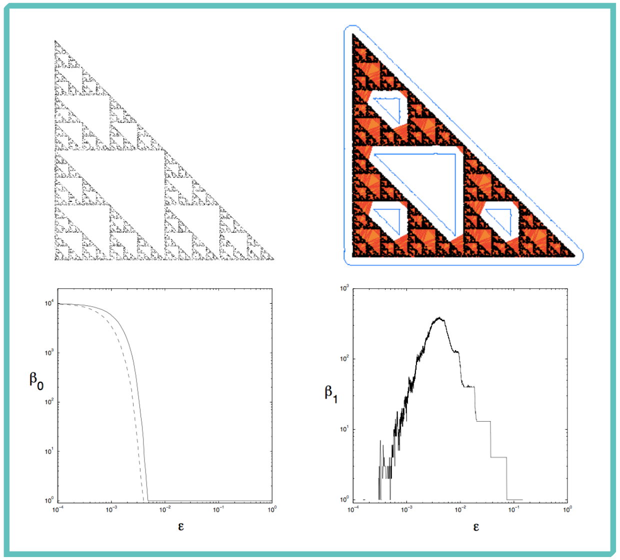


Image from [Robins 1999]

Persistent Homology

An Evolving Notion:

1990	1994	1999	2002
Frosini	Delfinado, Edelsbrunner	Robins	Edelsbrunner, Letscher, Zomorodian

Topological Persistence:

- Introduction and algebraic formulation of the notion of ***persistent homology***
- ***Description of an algorithm*** for computing persistent homology

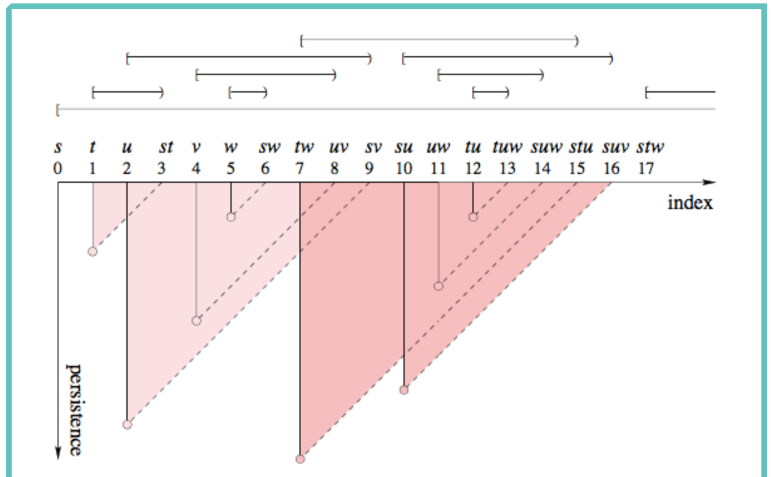


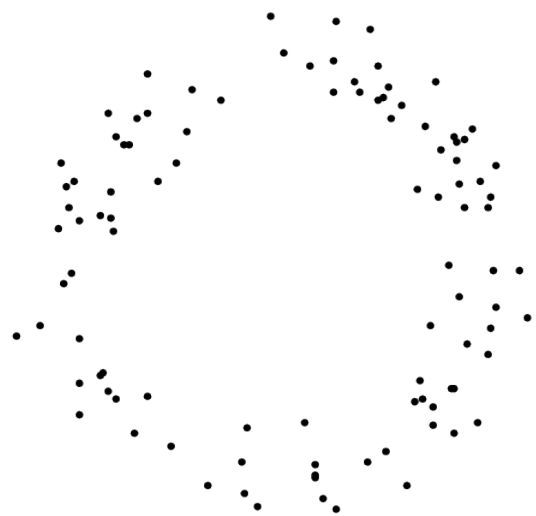
Image from [Edelsbrunner et al. 2002]

Persistent Homology

A Twofold Purpose:

Shape Description

- *Which is the shape of a given data?*

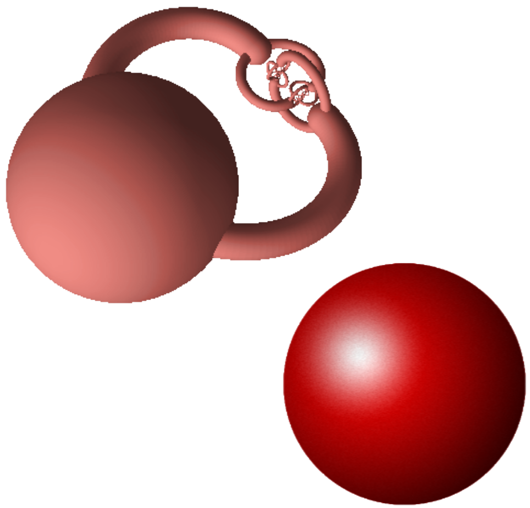


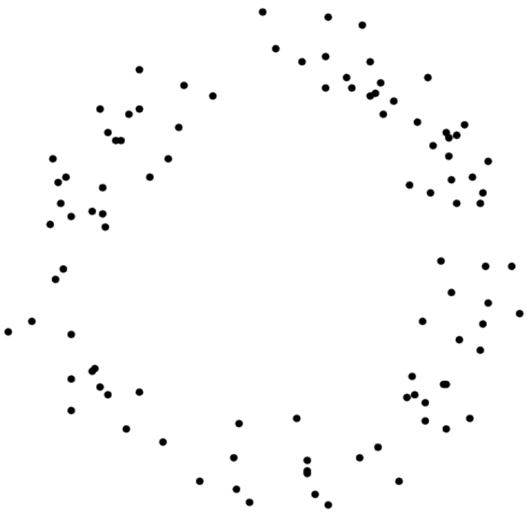
Persistent Homology

A Twofold Purpose:

Shape Description

- *Which is the shape of a given data?*





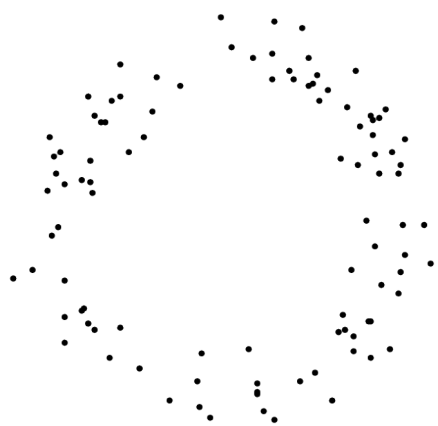
Shape Comparison

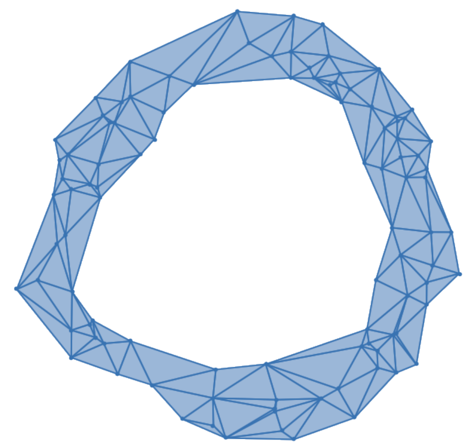
- *Given two data, do they have the same shape?*

Persistent Homology

Which is the shape of a given data?

Persistent homology allows for the retrieval of the ***"actual"*** homological information of a data





Point Cloud Dataset

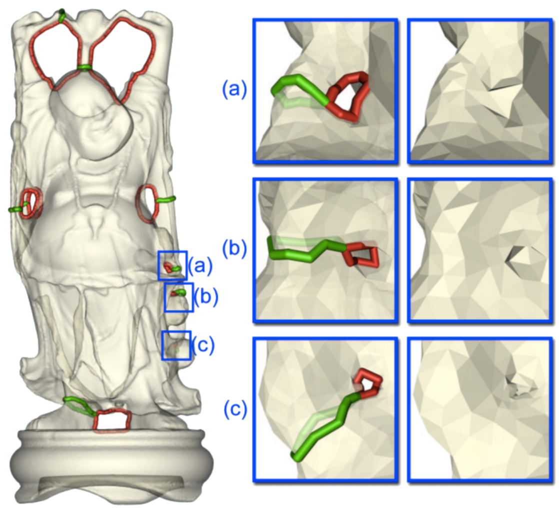
Topological Nature of the "Underlying" Shape

Image from [Bauer 2015]

Persistent Homology

✦ *Which is the shape of a given data?*

Persistent homology allows for the retrieval of the ***"actual"*** homological information of a data



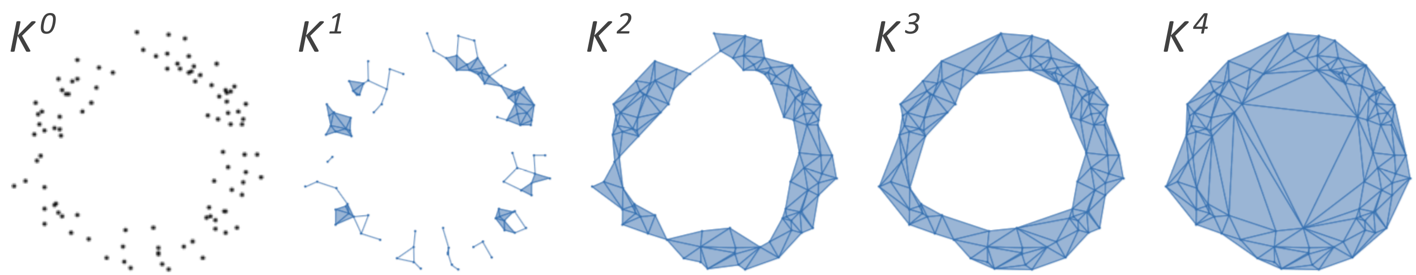
Noisy Dataset → *Relevant Homological Information*

Image from [Dey et al. 2008]

Persistent Homology

Definition:

A *filtration* $\mathscr{F}$ is a finite ***"growing" sequence*** of simplicial complexes



$$K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m$$

Most of the techniques transforming a dataset into a simplicial complex depending on the choice of a parameter actually produce a filtration

Persistent Homology

Definitions:

Given a filtration $\mathscr{F}$ and $p,q \in \mathbb{N}$ such that $0 \leq p \leq q \leq m$,

- the inclusion $K^p \subseteq K^q$ induces a *linear map* $i_k^{p,q} : H_k(K^p) \longrightarrow H_k(K^q)$
- the *(p,q)-persistent k-homology group* $H_k^{p,q}(\mathscr{F})$ of $\mathscr{F}$ is defined as

$$H_k^{p,q}(\mathcal{F}) := \mathrm{Im}(i_k^{p,q})$$

  and consists of the ***k-cycles of K^p that will not turn into k-boundaries in K^q***

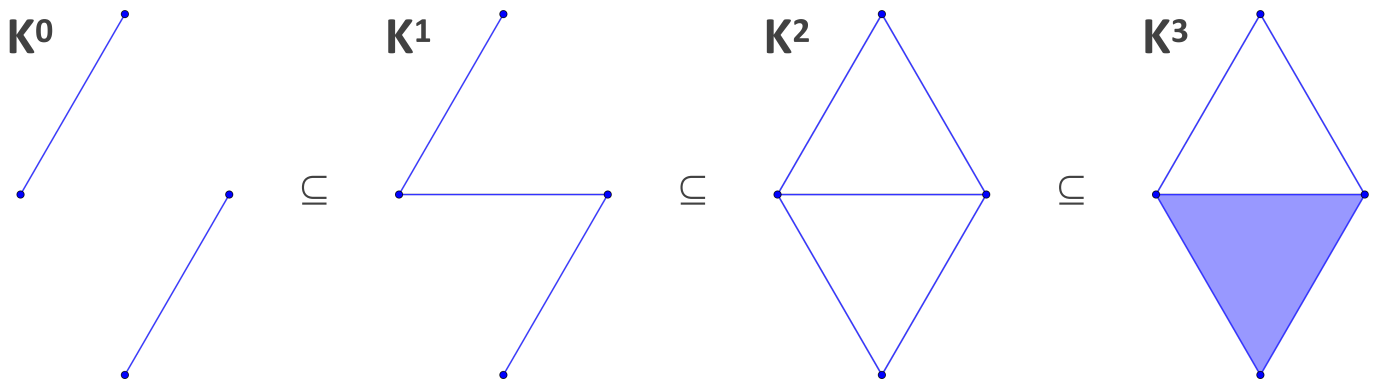
  (in other terms, it identifies the ***homology classes that "persist" from K^p to K^q***)

- the *(p,q)-persistent k^{th} Betti number* $\beta_k^{p,q}$ of $\mathscr{F}$ is defined as the *rank of $i_k^{p,q}$*

Persistent Homology

The ***core information*** of persistent homology is given by the ***persistence pairs***

Given a filtration $\mathscr{F}$: $K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m$,

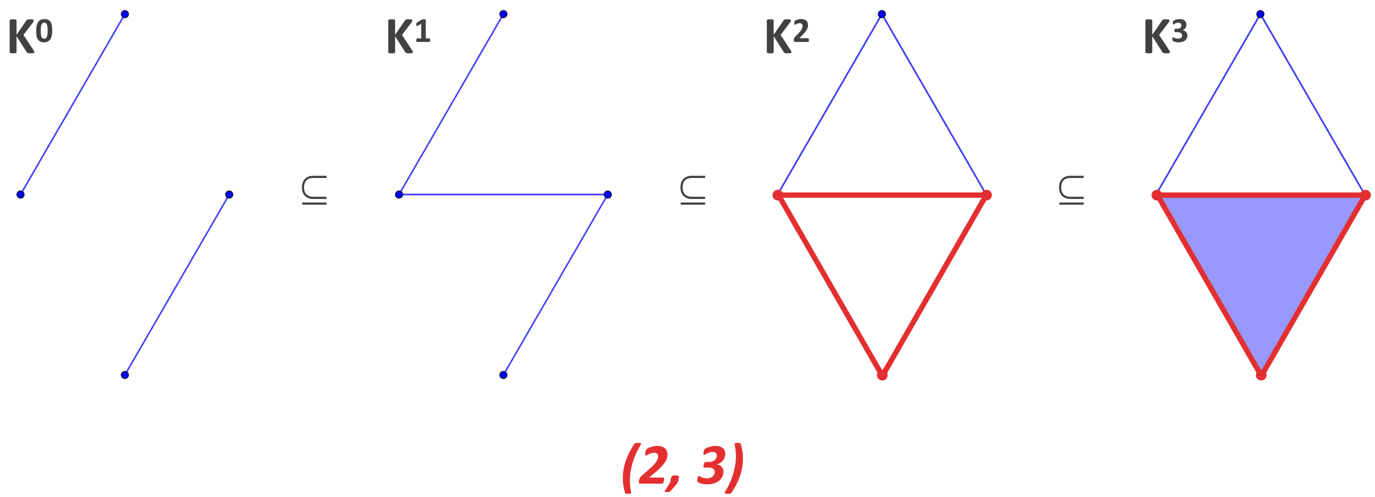


A persistence pair ***(p, q)*** is an element in $\{0, \ldots, m\} \times (\{0, \ldots, m\} \cup \{\infty\})$ such that $p < q$ representing a **homological class** that is **born at step *p*** and **dies at step *q***

Persistent Homology

The *core information* of persistent homology is given by the ***persistence pairs***

Given a filtration $\mathscr{F}$: $K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m$,

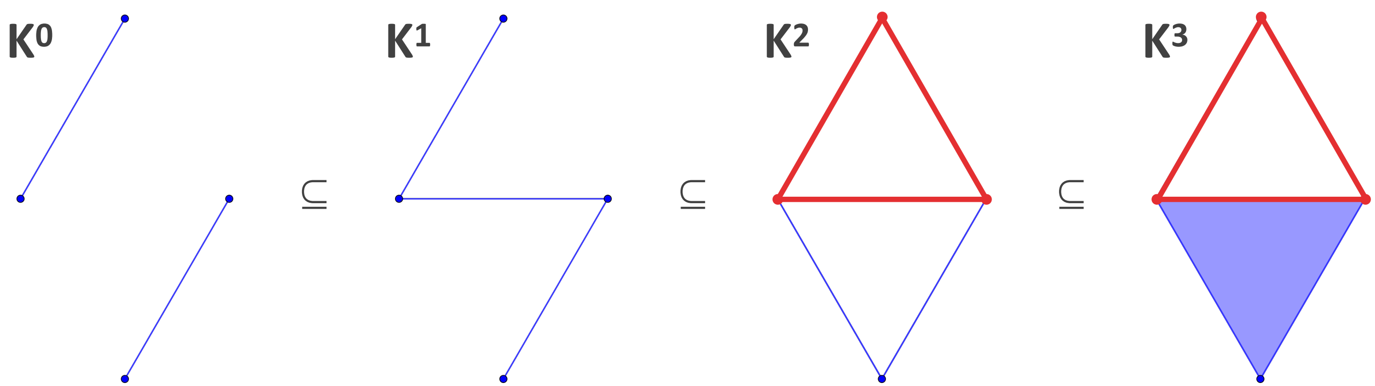


A persistence pair *(p, q)* is an element in $\{0, \ldots, m\} \times (\{0, \ldots, m\} \cup \{\infty\})$ such that $p < q$ representing a **homological class** that is **born at step *p*** and **dies at step *q***

Persistent Homology

The ***core information*** of persistent homology is given by the ***persistence pairs***

Given a filtration $\mathscr{F}$: $K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m$,



***(2, ∞)* essential pair**

A **persistence pair *(p, q)*** is an element in $\{0, \ldots, m\} \times (\{0, \ldots, m\} \cup \{\infty\})$ such that $p < q$ representing a **homological class** that is **born at step *p*** and **dies at step *q***

Persistent Homology

Given a filtration $\mathscr{F}$: $K^0 \subseteq K^1 \subseteq \dots \subseteq K^m$, $k \in \mathbb{N}$, and a field $\mathbb{F}$,

its ***persistence module*** $M := \bigoplus_{p \in \mathbb{N}} H_k(K^p; \mathbb{F})$ is a ***finitely generated $\mathbb{F}$[x]-module***

The corresponding structure theorem ensures us that

Theorem:

The persistence module M can be expressed as

$$M \cong \bigoplus_{k=1}^{t} \mathbb{F}[x](-p'_k) \oplus \bigoplus_{j=1}^{s} \left(\mathbb{F}[x]/(x^{q_j - p_j}) \right)(-p_j)$$

So, the persistence module M is completely determined by its persistence pairs

I.e., the collection of the pairs ***(p_i, q_i), (p'_j, ∞)***

Bibliography

General References:

- ***Books on TDA:***
  - A. J. Zomorodian. ***Topology for computing***. Cambridge University Press, 2005.
  - H. Edelsbrunner, J. Harer. ***Computational topology: an introduction***. American Mathematical Society, 2010.
  - R. W. Ghrist. ***Elementary applied topology***. Seattle: Createspace, 2014.
- ***Papers on TDA:***
  - G. Carlsson. ***Topology and data***. Bulletin of the American Mathematical Society 46.2, pages 255-308, 2009.

Today's References:

- ***Simplicial Homology:***
  - J. R. Munkres. ***Elements of algebraic topology***. CRC Press, 1984.
- ***Persistent Homology:***
  - U. Fugacci, S. Scaramuccia, F. Iuricich, L. De Floriani. ***Persistent homology: a step-by-step introduction for newcomers***. Eurographics Italian Chapter Conference, pages 1-10, 2016.